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ADMISSIBLE ESTIMATORS OF VARIANCE COMPONENTS IN NORMAL MIXED MODELS

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Abstract. A sufficient condition for an invariant quadratic estimator of a linear function of the vector of variance components to be admissible under the mean square error among all translation invariant estimators is given.

1. Introduction. Throughout the paper Y will stand for a random n-vector normally distributed with expectation $A\beta$ and covariance matrix $\sum_{i=1}^{p} \sigma_i V_i$, i.e., let

$$Y \sim N\Big(A\beta, \sum_{i=1}^{p} \sigma_i V_i\Big),$$

where A is a known $(n \times k)$-matrix, $V_1, \ldots, V_p$ are known nonnegative definite $(n \times n)$-matrices, while $\beta \in \mathscr{R}^k$ and $\sigma_1 \geqslant 0, \ldots, \sigma_p \geqslant 0$ are the unknown parameters. Assume that

$$\mathscr{R}(A) + \mathscr{R}\Big(\sum_{i=1}^{p} V_i\Big) = \mathscr{R}^n,$$

where $\mathscr{R}(\cdot)$ denotes the range of the matrix argument.

We concentrate on estimation of a linear function $F'\sigma$, where F' is the transpose of $(p \times s)$-matrix F $(s \leqslant p)$, while $\sigma = (\sigma_1, \ldots, \sigma_p)'$ is the vector of variance components. The regression vector is treated as a nuisance parameter.

We consider a class $\mathscr{I}_F$ of estimators based on MY, where M is the orthogonal projection matrix on the null space of A'. These estimators are invariant with respect to the translations $Y \to Y + A\beta$, $\beta \in \mathscr{R}^k$, and MY is a maximal invariant for this group of translations. Clearly, $MY \sim N(\boldsymbol{0}_n, M_\sigma)$, where $\boldsymbol{0}_n$ denotes the zero vector in $\mathscr{R}^n$, while

$$M_\sigma = \sum_{i=1}^{p} \sigma_i M_i, \quad M_i = MV_iM, \; i = 1, \ldots, p.$$

To compare estimators we shall use the mean square error defined for any estimator $\delta = \delta(MY)$ of $F'\sigma$ by

$$R(\delta, \sigma) = \mathrm{E}(\delta - F'\sigma)'(\delta - F'\sigma).$$

Let Θ be a subset of $\mathscr{R}^p$ defined by

$$\Theta = \{\sigma \in \mathscr{R}^p\colon\ \sigma \geqslant 0_p,\ \mathscr{R}(M_\sigma) = \mathscr{R}(M)\},$$

where the expression $\sigma \geqslant 0_p$ ($\sigma > 0_p$) means that all coordinates of σ are nonnegative (positive). Consider a subset $\mathscr{Q}_F \subset \mathscr{I}_F$ of the form

$$q_u = q_u(Y) = \frac{Y'M_u^+ Y}{2+r} F'u, \quad u \in \Theta, \tag{1.1}$$

where $r = \operatorname{rank}(M)$, while M_u^+ denotes the Moore–Penrose g-inverse of M_u. The estimators in $\mathscr{Q}_F$ have the following property. For a given $u \in \Theta$ the estimator q_u minimizes the risk at each point $\sigma = \lambda u$, $\lambda > 0$, among all invariant quadratic estimators, i.e., among estimators of the form

$$(Y'MA_1MY, \ldots, Y'MA_sMY)',$$

where $A_1, \ldots, A_s$ can be arbitrary symmetric $(n \times n)$-matrices.

Note that if $M_1, \ldots, M_p$ commute, as in the case of balanced models, then there exist idempotent nonzero matrices $Q_1, \ldots, Q_m$, say, with their ranges contained in $\mathscr{R}(M)$, such that Q_iQ_j is zero matrix for $i \neq j = 1, \ldots, m$, and that

$$M_i = \sum_{j=1}^{m} h_{ij} Q_j, \quad i = 1, \ldots, p.$$

In this case M_u^+ can be represented as

$$M_u^+ = \sum_{j=1}^{m} (1/\theta_j) Q_j,$$

where $(\theta_1, \ldots, \theta_m)' = H'u$, while $H = (h_{ij})$.

Karlin [3] has proved that for $p = 1$ the set $\mathscr{Q}_F$, $F \in \mathscr{R}$, contains exactly one estimator, which is the only invariant quadratic estimator admissible for σ among $\mathscr{I}_F$. For $p > 1$ and under the assumption that matrices $M_1, \ldots, M_p$ commute Farrell et al. [2] have shown that each estimator in $\mathscr{Q}_F$ is admissible among $\mathscr{I}_F$. Moreover, they have also proved that $\mathscr{Q}_I$, where I denotes the identity $(p \times p)$-matrix, represents the class of all invariant quadratic estimators admissible for σ among $\mathscr{I}_I$. Dey and Gelfand [1] have established the admissibility of estimators in $\mathscr{Q}_F$, $F \in \mathscr{R}^p$, under more restrictive conditions.

In this paper we drop the assumption that matrices $M_1, \ldots, M_p$ commute and prove that each estimator in a subset $\mathscr{Q}_F^*$ of $\mathscr{Q}_F$ consisting of q_u with $u > 0_p$ is admissible for $F'\sigma$ among $\mathscr{I}_F$.

2. Results. We shall use an idea of Farrell et al. [2] to establish the admissibility of estimators in $\mathscr{Q}_F^*$ also in the case where matrices $M_1, \ldots, M_p$ do not commute.

THEOREM. *All estimators in $\mathscr{Q}_F^*$ are admissible for $F'\sigma$ among the class $\mathscr{I}_F$ of invariant estimators.*

Proof. According to a lemma due to Shinozaki (see, e.g., [4]) it is sufficient to prove the theorem for $F = I$.

First note that since

$$M_\sigma M_u^+ M_\sigma M_u^+ M_\sigma = \frac{\lambda}{2} M_\sigma M_u^+ M_\sigma$$

for $\sigma = \sigma_\lambda = (\lambda/2)u$, $\lambda > 0$, and since $\operatorname{rank}(M_u) = r$ for $u > 0_p$, it follows that when $\sigma = \sigma_\lambda$, the random variable $Y' M_u^+ Y$ has the gamma distribution with the shape parameter $r/2$ and the scale parameter λ. Thus, by Karlin's theorem,

$$q = \frac{2}{2+r} Y' M_u^+ Y$$

is admissible for λ among all estimators based on $Y' M_u^+ Y$.

The risk of any estimator $\delta = (\delta_1, \ldots, \delta_p)'$ at σ_λ can be written as

$$R(\delta, \sigma_\lambda) = \frac{1}{4} \operatorname{E} \sum_{i=1}^{p} (2\delta_i - \lambda u_i)^2 = \frac{a}{4} \operatorname{E} \left[\sum_{i=1}^{p} \frac{u_i^2}{a} \left(\frac{2\delta_i}{u_i} - \lambda \right)^2 \right],$$

where $a = \sum_{i=1}^{p} u_i^2$. Applying Jensen's inequality to the expression in brackets, we obtain the inequality

$$R(\delta, \sigma_\lambda) \geqslant \frac{a}{4} \operatorname{E} \left(\frac{2}{a} \sum_{i=1}^{p} u_i \delta_i - \lambda \right)^2$$

which is strict unless $\delta_i/u_i = \delta_j/u_j$ for all $i, j = 1, \ldots, p$.

Since the random variable $Y' M_u^+ Y$ is a sufficient statistics for λ when $\sigma = \sigma_\lambda$, there exists an estimator δ^* of λ based on $Y' M_u^+ Y$ as good as $2a^{-1} \sum_{i=1}^{p} u_i \delta_i$. Moreover, since, as we have already noted, q is admissible for λ and since the mean square error of q_u and q are related at $\sigma = \sigma_\lambda$ by

$$R(q_u, \sigma_\lambda) = \frac{a}{4} R(q, \sigma_\lambda),$$

it follows that if, say, δ dominates q_u, then

$$R(q, \lambda) = \operatorname{E} \left(\frac{2}{a} \sum_{i=1}^{p} u_i \delta_i - \lambda \right)^2.$$

Consequently, $\delta_i = u_i q$ for all i with probability 1, so that $\delta = q_u$ with probability 1. But this contradicts the assumption that δ dominates q_u and concludes the proof of the Theorem.

It is an open problem whether there exist alternative invariant quadratic estimators to (1.1) admissible for σ in the case where matrices $M_1, \ldots, M_p$ do not commute.

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